UK INTERMEDIATE MATHEMATICAL CHALLENGE

THURSDAY 7th FEBRUARY 2008

Organised by the **United Kingdom Mathematics Trust**
from the School of Mathematics, University of Leeds

http://www.ukmt.org.uk

The Actuarial Profession
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SOLUTIONS LEAFLET

This solutions leaflet for the IMC is sent in the hope that it might provide all concerned with some alternative solutions to the ones they have obtained. It is not intended to be definitive. The organisers would be very pleased to receive alternatives created by candidates.

1. E The clocks do not go forward or back this week, so there are seven 24-hour days, that is 168 hours.

2. C $2 + 3 + 5 \times 7 = 5 + 35 = 40$. As $40 = 2^3 \times 5$, the largest prime number which divides exactly into it is 5.

3. B $0{\cdot}75 \div \frac{3}{4} = \frac{3}{4} \div \frac{3}{4} = 1$.

4. B The large square is made up of 25 small squares, 15 of which are shaded. So $\frac{15}{25}$ of the large square is shaded, corresponding to $\frac{15}{25} \times 100\% = 60\%$.

5. D In each of the fractions, the denominator is 18 and the sum of the digits of the numerator is also 18. So every numerator is a multiple of 9 and the even numerators are also multiples of 18. However, 873 is not a multiple of 18, so $\frac{873}{8 + 7 + 3}$ is the only expression not equal to a whole number.

6. B Let shape C have width 1 unit, height 1 unit and depth 1 unit. Then the volumes, in units³, of the five shapes are: A 2, B $2\frac{1}{2}$, C $\frac{1}{2}$, D $1\frac{1}{2}$, E 4. These total $10\frac{1}{2}$ units³, so we may deduce that the cube formed by the four shapes will have side 2 units and volume 8 units³. Hence B is the shape which is not required. The cube may be formed by placing C next to D to form a shape identical to A. This combination is then placed alongside A to form a shape identical to E. If shape E is now rotated through 180° about a suitable axis, it may be placed with the combination of shapes A, C and D to form a cube.

7. D Let the original number be x. Then $x^2 = 0{\cdot}7x$, that is $x\left(x - 0{\cdot}7\right) = 0$. So $x = 0$ or $x = 0{\cdot}7$, but as x is non-zero it is $0{\cdot}7$.

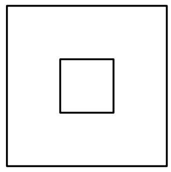
8. A October has 31 days so, in any year, in October there are three days of the week which occur five times and four days which occur four times. As there were four Tuesdays and four Fridays, there could not have been five Wednesdays or five Thursdays, so the days which occurred five times were Saturday, Sunday and Monday. Hence October 1st fell on a Saturday, which means that October 31st was a Monday.

9. D Let the smaller cubes have side of length 1 unit. So the original cube had side of length 3 units and hence a surface area of 54 units², all of which was painted blue. The total surface area of the 27 small cubes is 27×6 units², that is 162 units². So the required fraction is $\frac{54}{162} = \frac{1}{3}$.

10. C In every triangle the length of the longest side is less than the sum of the lengths of the two other sides. So if the triangle has sides of length 5 cm and 6 cm, then the length of the third side is greater than 1 cm, but less than 11 cm. Hence the perimeter, p cm, of the triangle satisfies $12 < p < 22$. So 15 cm is the only one of Perry's suggested values which could be correct.

11. E $S = 25\%$ of $60 = 15$. $U = \dfrac{60}{0{\cdot}8} = 75$. $M = \dfrac{80}{0{\cdot}25} = 320$. So $S + U + M = 410$.

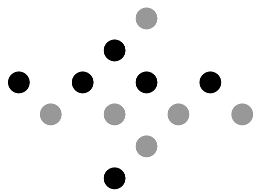
12. C In each of 6 possible directions, the view of the sculpture is as shown, with the outer square having side 3 units and the inner square having side 1 unit.
So the surface area of the sculpture is $6 \times 8\,\text{units}^2 = 48\,\text{units}^2$.



13. D The sum of all 64 numbers is $64 \times 64 = 64^2$. The sum of the first 36 numbers is $36 \times 36 = 36^2$. So the sum of the remaining 28 numbers is $64^2 - 36^2 = (64 + 36)(64 - 36) = 2800$. Therefore the mean of these 28 numbers is 100.

14. B Let A and B denote the ends of the first length of rope and C and D denote the ends of the second length of rope. Then Pat chooses one of 6 different possible combinations: (A, B), (A, C), (A, D), (B, C), (B, D), (C, D). Sam now holds one untied length of rope and one tied loop of rope if, and only if, Pat has chosen (A, B) or (C, D) so the required probability is $\dfrac{2}{6} = \dfrac{1}{3}$.
(*Alternatively: Irrespective of whichever end Pat chooses first, Sam will hold one untied length of rope and one tied loop of rope if, and only if, Pat now chooses a particular one of the three remaining ends, namely the other end of the same rope. As each of the three ends is equally likely to be chosen, the required probability is* $\frac{1}{3}$.)

15. A Notice that a single copy of the logo consists of four dots which lie in a straight line plus two other dots which lie on the perpendicular bisector of this line. These two dots are not evenly spaced above and below the line of four dots. Of the options given, only A has two lines of four dots with two more dots in the correct positions relative to each line.



16. D The problem may be solved by firstly calculating the third and fourth terms of the sequence, but an algebraic method does reduce the amount of calculation involved. Let the first two terms be x and y respectively. Then the third term is $\frac{1}{2}(x + y)$, whilst the fourth term is $\frac{1}{4}(x + 3y)$. So the fifth term is $\frac{1}{8}(3x + 5y)$.
Putting $x = \dfrac{2}{3}$ and $y = \dfrac{4}{5}$, we obtain $\dfrac{2 + 4}{8} = \dfrac{3}{4}$.

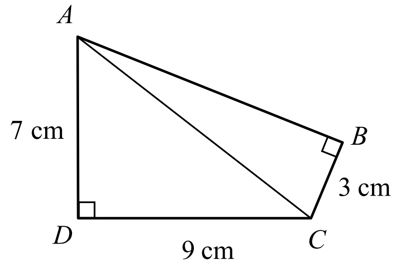
17. D As the perimeter of the square has length 8, the square has side length 2. So the diameter of each of the circles is 1. The perimeter of the shaded region consists of four semi-circular arcs and four quarter-circle arcs, so it has length equal to three times the circumference of one circle, that is 3π.

18. C The five options are: A 0.20088888… B 0.20080000… C 0.20080808… D 0.20080080… E 0.20082008….
So in ascending order they are B D C E A.

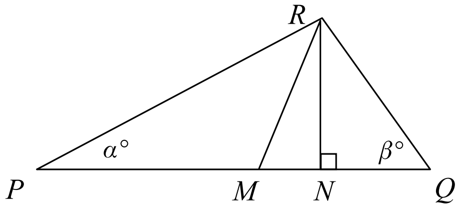
19. E Using 'the difference of two squares':

$$(1+x+y)^2-(1-x-y)^2=(1+x+y+1-x-y)(1+x+y-1+x+y)=2(2x+2y)=4(x+y).$$

20. A Let the vertices of the quadrilateral be A, B, C, D as shown. Then, by Pythagoras' Theorem: $AC^2 = AD^2 + DC^2 = (7^2+9^2)\,\text{cm}^2 = 130\,\text{cm}^2$. Similarly, $AB^2 + BC^2 = AC^2$, so $AB^2 = (130-9)\,\text{cm}^2 = 121\,\text{cm}^2$. Therefore AB has length 11 cm and the area, in cm², of quadrilateral $ABCD$ is $\frac{1}{2}\times 9\times 7+\frac{1}{2}\times 3\times 11 = 48$.



21. B The sum of the interior angles of a triangle is 180°, so $\angle PRQ = (180-\alpha-\beta)°$; hence $\angle QRM = \left(90-\frac{\alpha}{2}-\frac{\beta}{2}\right)°$. As RN is perpendicular to PQ, $\angle NRQ = (90-\beta)°$. So $\angle MRN = \left(90-\frac{\alpha}{2}-\frac{\beta}{2}\right)°-(90-\beta)° = \left(\frac{\beta}{2}-\frac{\alpha}{2}\right)°$.

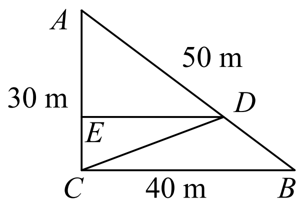


22. A Both £4.20 and £7.70 are multiples of 70p, so £C must also be a multiple of 70p. Of the options given, only £91 is a multiple of 70p, but it remains to check that a total cost of £91 is possible. If there are 7 children and 8 adults, then the total cost is $7\times £4.20+8\times £7.70 = £29.40+£61.60 = £91$.

23. E The only digits which will appear the same when reflected in the glass table-top are 0, 1, 3 and 8. So it is necessary to find the number of times in a 24-hour period that the display on the clock is made up only of some or all of these four digits. The first of the digits, therefore, may be 0 or 1; the second digit may be 0, 1, 3 or 8; the third digit may be 0, 1 or 3; the fourth digit may be 0, 1, 3 or 8. So the required number is $2\times 4\times 3\times 4 = 96$.

24. C As the figure has rotational symmetry of order 4, $ABEF$ is a square.
Area $ABEF = 4\times$ area $\Delta BDA = 4\times\frac{1}{2}BD\times DA = 2BD^2 = 24\,\text{cm}^2$ so $BD=\sqrt{12}$ cm $= 2\sqrt{3}$ cm. As $ABEF$ is a square, $\angle ABD = 45°$ so $\angle CBD = 45°-15° = 30°$.
Therefore $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{CD}{BD} = \dfrac{CD}{2\sqrt{3}}$, so $CD = 2$ cm.

25. E In the diagram on the right, triangle ABC represents the garden, CD represents the fence and E is the foot of the perpendicular from D to AC.



The two sections of the garden have the same perimeter so AD is 10 m longer than DB. Hence $AD = 30$ m and $DB = 20$ m. As $\angle AED$ and $\angle ACB$ are both right angles, triangles AED and ACB are similar.
So $\frac{AE}{AC} = \frac{AD}{AB} = \frac{30}{50}$. Hence $AE = \frac{3}{5}\times 30\,\text{m} = 18\,\text{m}$. So $EC = (30-18)\,\text{m} = 12\,\text{m}$.
Also, $\frac{ED}{CB} = \frac{AD}{AB} = \frac{30}{50}$. Hence $ED = \frac{3}{5}\times 40\,\text{m} = 24\,\text{m}$.
Finally, by Pythagoras' Theorem: $CD^2 = EC^2+ED^2 = (12^2+24^2)\,\text{m}^2 = 5\times 12^2\,\text{m}^2$.
So the length of the fence is $12\sqrt{5}$ m.